CALCULUS.

Mixed Exercise.

1. Find $\frac{dy}{dx}$ if $y = (x^2 + 1)^2$.

$$y = (x^2 + 1)^2$$

$$y = (x^2 + 1)(x^2 + 1)$$

$$y = (x^2)^2 + 2(x^2)(1) + (1)^2$$

$$y = x^4 + 2x^2 + 1$$

$$\frac{dy}{dx} = 4x^3 + 4x.$$

2. Find the equation of the tangent to the curve $y = 6x^2 - x - 4$ at the point $A(1,1)$.

Curve; $y = 6x^2 - x - 4$ **at** $A(1,1)$

Gradient of tangent;

$$\frac{dy}{dx} = 12x - 1$$

at $x = 1$

$$M_1 = 12(1) - 1$$

$$= 11$$

For the tangent $M_1 = M_2$

$$= 11$$

By taking Point $A(1,1)$

$$M_2 = 11$$

and Point (x, y)

$$\frac{y-1}{x-1} = 11$$

$$y = 11x - 11 + 1$$

$$y = 11x - 10.$$

3. Find the equation of the tangent to the curve $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + x - 2$ at a point where $x = 1$. Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

Curve; $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + x - 2$

at $x = 1.$

Gradient of tangent;

$$\frac{dy}{dx} = x^2 - 3x + 1$$

at $x = 1$

$$M_1 = (1)^2 - 3(1) + 1$$

$$= -1$$

$$y = \frac{1}{3}(1)^3 - \frac{3}{2}(1)^2 + (1) - 2$$

$$y = -\frac{13}{6}$$

Point of tanging $\left(1, -\frac{13}{6}\right)$

Equation;

$$\frac{y + \frac{13}{6}}{x - 1} = -1$$

$$y = -x + 1 - \frac{13}{6}$$

$$6y = -6x - 7$$

$$6y + 6x + 7 = 0.$$

4. The gradient of the tangent to the curve $y = ax^3 + bx^2$ at the point $A(-1, -5)$ is 12. Find the value a and b.

$$y = ax^3 + bx^2$$

at $A(-1, -5)$

$x = -1$ **and** $y = -5$

$-a + 5 = -5 \ldots\ldots$ **eq. 1**

$$\frac{dy}{dx} = 3ax^2 + 2bx = 12$$

at $x = -1$

$3a - 2b = 12 \ldots\ldots\ldots$ **eq. 2**

From eq. 1

$$b = a - 5$$

substituting in eq. 2,

we have;

$$3a - 2(a - 5) = 12$$

$$3a - 2a + 10 = 12$$

$$a = 12 - 10$$

$$a = 2$$

$$b = 2 - 5$$

$$b = -3$$

$$a = 2$$

$$b = -3$$

5. Find the equation of the normal to the curve $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x + 8$ at its $y-$intercept.

$$y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x + 8$$

$$\text{at } y - \text{intercept}, x = 0$$

$$y = \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 + 3(0) + 8$$

$$y = 8$$

Point of tanging A(0, 8)

Gradient of the tangent;

$$\frac{dy}{dx} = x^2 - x + 3$$

$$\text{at } x = 0$$

$$M_1 = (0)^2 - (0) + 3$$

$$M_1 = 3$$

For normal;

$$M_1M_2 = -1$$

$$M_2 = -\frac{1}{3}$$

Taking Point;

$$A(0, 8), (x, y) \text{ and } M_2 = -\frac{1}{3}$$

$$\frac{y - 8}{x - 0} = -\frac{1}{3}$$

$$y = -\frac{1}{3}x + 8$$

$$3y = -x + 24.$$

6. The gradient of the of the tangent to the curve $y = ax^3 + \frac{1}{2}x^2 - 3x + 7$ at a point where $x = 2$ is 5. Find the value of a.

$$y = ax^3 + \frac{1}{2}x^2 - 3x + 7$$

$$\text{at } x = 2, \frac{dy}{dx} = 5$$

$$\frac{dy}{dx} = 3ax^2 + x - 3$$

$$\text{at } x = 2;$$

$$3a(2)^2 + (2) - 3 = 5$$

$$12a - 1 = 5$$

$$12a = 6$$

$$a = \frac{1}{2}.$$

7. Find the equation of the normal to the curve $y = \frac{x(9x^2-1)}{3x+1}$ at a point where x = 2.

$$y = \frac{x(9x^2-1)}{3x+1}$$

$$y = \frac{x(3x-1)(3x+1)}{(3x+1)}$$

$$y = x(3x-1)$$

$$y = 3x^2 - x$$

$$\text{at } x = 2$$

$$y = 3(2)^2 - (2)$$

$$y = 10$$

Point of tanging;

$$A(2, 10)$$

Gradient of Tangent;

$$\frac{dy}{dx} = 6x - 1$$

$$\text{at } x = 2$$

$$M_1 = 6(2) - 1$$

$$= 11$$

Gradient of normal;

$$M_1M_2 = -1$$

$$M_2 = -\frac{1}{11}$$

Equation;

$$\frac{y-10}{x-2} = -\frac{1}{11}$$

$$11y - 110 = -x + 2$$

$$11y + x - 112 = 0.$$

8. Determine the point on the curve $y = \frac{1}{2}x^2 + 4$ at which the gradient is 8 hence find the equation of the normal to the curve at this point.

$$y = \frac{1}{2}x^2 + 4$$

$$\frac{dy}{dx} = x$$

$$x = 8$$

$$y = \frac{1}{2}(8)^2 + 4$$

$$= 36$$

Point of tanging;

$$A(8, 36)$$

Gradient of normal;

$$M_1M_2 = -1$$

$$M_2 = -\frac{1}{8}$$

Eqaution;

$$\frac{y-36}{x-8} = -\frac{1}{8}$$

$$8y - 288 = -x + 8$$

$$8y + x - 296 = 0.$$

9. Find the equation of the tangent to the curve $y = x^2 - 3x + 7$ at a point where the gradient is 5.

$$y = x^2 - 3x + 7$$

$$\frac{dy}{dx} = 2x - 3$$

$$= 5$$

$$2x - 3 = 5$$

$$2x = 8$$

$$x = 4$$

$$y = (4)^2 - 3(4) + 7$$

$$= 11$$

Point of tanging;

$$A(4, 11)$$

Equation;

$$\frac{y - 11}{x - 4} = 5$$

$$y = 5x - 20 + 11$$

$$y = 5x - 9.$$

10. Find the equation of the tangents to the curve $y = 5x - x^2$ at the points where it intersect with the line $y = x$.

$$y = 5x - x^2 \ldots\ldots \text{eq. } 1$$

$$y = x \ldots\ldots \text{eq. } 2$$

Substituting eq. 2 into eq. 1

$$5x - x^2 = x$$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0$$

or

$$x - 4 = 0$$

$$x = 4$$

when $x = 0$;

$$y = 5(0) - (0)^2$$

$$y = 0$$

Point of tanging;

$$A(0, 0)$$

Gradient of the tangent;

$$\frac{dy}{dx} = 5 - 2x$$

at $x = 0$

$$\frac{dy}{dx} = 5 - 2(0)$$

$$= 5$$

Equation;

$$\frac{y - 0}{x - 0} = 5$$

$$y = 5x \ldots\ldots 1^{st} \text{ eaquation.}$$

or;

at $x = 4$

$$y = 5(4) - (4)^2$$

$$= 4$$

Point of tanging;

$$B(4, 4)$$

Gradient of the tangent;

$$\frac{dy}{dx} = 5 - 2x$$

at $x = 4$

$$\frac{dy}{dx} = 5 - 2(4)$$

$$= -3$$

Equation;

$$\frac{y - 4}{x - 4} = -3$$

$$y = -3x + 12 + 4$$

$$y = -3x + 16 \ldots. 2^{nd} \text{ eaquation.}$$

11. Find the coordinates of the stationary points of the curve $y = -x^3 + 3x^2 + 9x - 6$.

$$\mathbf{y = -x^3 + 3x^2 + 9x - 6}$$

At stationary point;

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

$$-3x^2 + 6x + 9 = 0$$

$$x = \frac{-6 \pm \sqrt{(-6)^2 - 4(-3)(9)}}{2(-3)}$$

$$x = \frac{-6 \pm 12}{-6}$$

$$x = -1 \text{ or } x = 3$$

$$\text{when } x = -1$$

$$y = -(-1)^3 + 3(-1)^2 + 9(-1) - 6$$

$$= -11$$

$$\text{Turning point } (-1, -11)$$

$$\text{when } x = 3$$

$$y = -(3)^3 + 3(3)^2 + 9(3) - 6$$

$$= 21$$

$$\text{Turning point } (3, 21).$$

12. Find the coordinates of the stationary points of the curve $y = 2x^3 - 3x^2 - 36x - 8$.

$$y = 2x^3 - 3x^2 - 36x - 8$$

$$\text{At stationary point } \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 6x^2 - 6x - 36$$

$$6x^2 - 6x - 36 = 0$$

$$x^2 - x - 6 = 0$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)}$$

$$x = \frac{1 \pm 5}{2}$$

$$x = 3 \text{ or } x = -2$$

$$\text{when } x = 3$$

$$y = 6(3)^2 - 6(3) - 36$$

$$= -89$$

$$\text{Turnimg point } (3, -89)$$

$$\text{when } x = -2$$

$$y = 6(-2)^2 - 6(-2) - 36$$

$$= 36$$

$$\text{Turnimg point } (-2, 36)$$

13. Given that $y = x^3 - 4x$, find;

(a) The intercepts of the curve.

$$y = x^3 - 4x$$

$$\text{at y} - \text{intercept}, x = 0$$

$$y = (0)^3 - 4(0)$$

$$y = 0$$

$$\text{Coordinates } (0, 0)$$

$$\text{at x} - \text{intercept}, y = 0$$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x = 0$$

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \sqrt{4}$$

$$x = \pm 2$$

$$\text{Coordinates } (-2, 0) \text{ or } (2, 0)$$

(b) The stationary points of the curve.

$$\text{at stationary point } \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 3x^2 - 4$$

$$3x^2 - 4 = 0$$

$$3x^2 = 4$$

$$x^2 = \frac{4}{3}$$

$$x = \sqrt{\frac{4}{3}}$$

$$x = \pm\frac{2}{\sqrt{3}}$$

$$\text{when } x = \frac{2}{\sqrt{3}}$$

$$y = \left(\frac{2}{\sqrt{3}}\right)^3 - 4\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{8}{3\sqrt{3}} - 4\left(\frac{2}{\sqrt{3}}\right)$$

$$= 1.5396$$

$$\text{Turning point } (1.1547, 1.5396)$$

$$\text{when } x = -\frac{2}{\sqrt{3}}$$

$$y = \left(-\frac{2}{\sqrt{3}}\right)^3 - 4\left(-\frac{2}{\sqrt{3}}\right)$$

$$= -\frac{8}{3\sqrt{3}} + 4\left(\frac{2}{\sqrt{3}}\right)$$

$$= 3.0792$$

$$\text{Turning point } (-1.1547, 3.0792)$$

14. Investigate the stationary points for the curve $y = 2x^3 + 5$.

$$y = 2x^3 + 5$$

At stationary point;

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 6x^2$$

$$6x^2 = 0$$

$$x^2 = 0$$

$$x = \pm 0$$

At $x = 0,$

$$y = 2(0)^3 + 5$$

$$y = 0$$

Coordinates $(0, 5)$

Nature of turning point;

x	−1	0	1
$\frac{dy}{dx} = 6x^2$	6	0	6
Nature			

Point $(0, 5)$ **is a point of inflection**

15. Identify the stationary points of the curve $y = 2x^2 - x^4 + 2$.

$$y = 2x^2 - x^4 + 2$$

At stationary point;

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 4x - 4x^3$$

$$4x - 4x^3 = 0$$

$$4x(1 - x^2) = 0$$

$$4x(1 - x)(1 + x) = 0$$

Either $x = 0$ **or** $x = -1$ **or** $x = 1$

at $x = 0$

$$y = 2(0)^2 - (0)^4 + 2$$

$$y = 2$$

Turning point $(0, 2)$

At $x = -1$

$$y = 2(-1)^2 - (-1)^4 + 2$$

$$= 3$$

Turning point $(-1, 3)$

At $x = 1$

$$y = 2(1)^2 - (1)^4 + 2$$

$$= 1$$

Turning point $(1, 3)$.

16. The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 8x + 2$. If the curve passes through a point (2,5), find its equation.

$$\frac{dy}{dx} = 3x^2 - 8x + 2$$

$$y = \int (3x^2 - 8x + 2)dx$$

$$y = x^3 - 4x^2 + 2x + c$$

$$\text{At } (2, 5)$$

$$5 = (2)^3 - 4(2)^2 + 2(2) + c$$

$$c = 5 + 4$$

$$c = 9$$

$$y = x^3 - 4x^2 + 2x + 5.$$

17. The gradient of a curve is given by $\frac{dy}{dx} = 4x - 2$. If the minimum value of the curve is 7, find its equation.

$$\frac{dy}{dx} = 4x - 2$$

$$y = \int (4x - 2)dx$$

$$y = 2x^2 - 2x + c$$

$$\text{At minimu point } \frac{dy}{dx} = 0$$

$$4x - 2 = 0$$

$$4x = 2$$

$$x = \frac{1}{2}$$

$$\text{at } x = \frac{1}{2}, y = 7$$

$$7 = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + c$$

$$7 = -\frac{1}{2} + c$$

$$c = 7\frac{1}{2}$$

$$y = 2x^2 - 2x + 7\frac{1}{2}.$$

18. Evaluate $\int_{t=2}^{4}(3t^2+4t+10)dt$.

$$\int_{t=4}^{4}(t^3+2t^2+10t)dt$$

$$= [t^3+2t^2+10]_2^4$$

$$= [(4)^3+2(4)^2+10(4)] - [(2)^3+2(2)^2+10(2)]$$

$$= 136 - 36$$

$$= 100 \text{ square units.}$$

19. Find the value of a if $\int_{a}^{3}(2x+4)\,dx = 25$.

$$\int_{a}^{3}(2x+4)\,dx = 25$$

$$[x^2+4x]_a^3 = 25$$

$$[(3)^2+4(3)] - [(a)^2+4(a)] = 25$$

$$21 - a^2 - 4a = 25$$

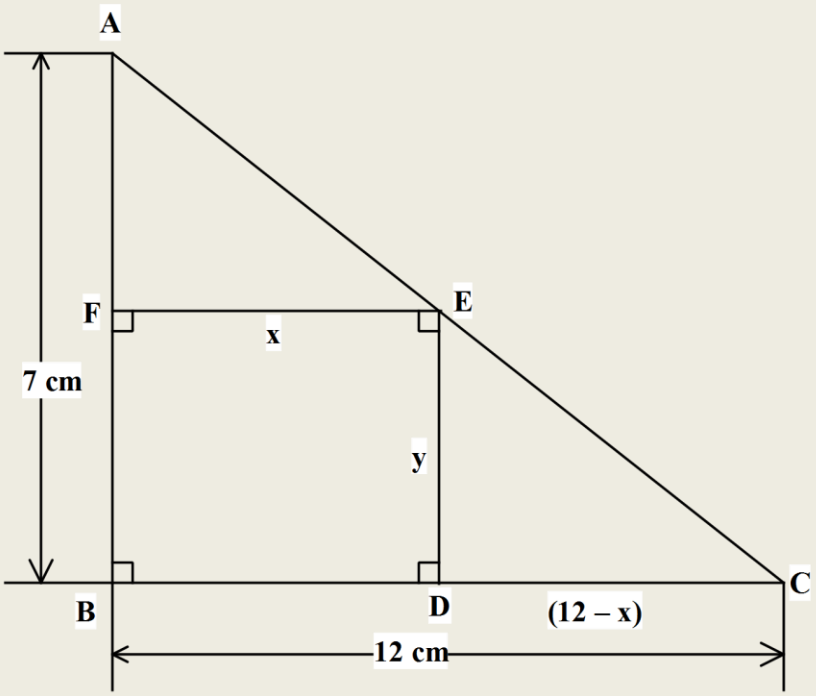
$$a^2+4a+4 = 0$$

$$a = \frac{-4 \pm \sqrt{(4)^2-4(1)(4)}}{2(1)}$$

$$a = \frac{-4 \pm 0}{2}$$

$$= -2.$$

20. A rectangle with maximum possible area is inscribed in a right angle triangle with base 12 cm and height 7 cm. Determine the dimensions of the rectangle.



Cosider ΔABC and ΔEDC

$$\frac{AB}{ED} = \frac{BC}{DC}$$

$$\frac{7}{y} = \frac{12}{(12 - x)}$$

$$12y = 84 - 7x$$

$$y = 7 - \frac{7}{12}x \ldots\ldots\ldots eq.1$$

But area of rec. FEDB

$$A = xy$$

$$A = x\left(7 - \frac{7}{12}x\right)$$

$$A = 7x - \frac{7}{12}x^2.$$

$$\text{At max. area } \frac{dA}{dx} = 0$$

$$7 - \frac{7}{6}x = 0$$

$$\frac{7}{6}x = 7$$

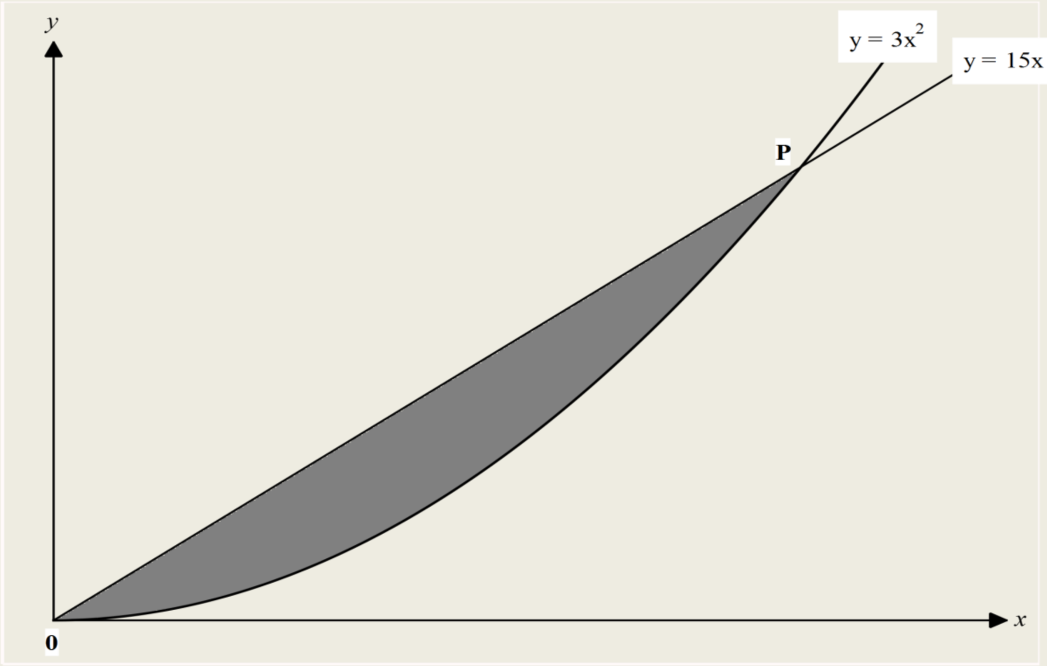
$$x = 6 \text{ cm}$$

$$y = 7 - \frac{7}{12}(6)$$

$$y = 3.5 \text{ cm}.$$

Area Under a Curve.

1. In the figure below, the shaded region is bounded between the line $y = 15x$ and the curve $y = 3x^2$.



(a) Determine the coordinates of P. (3mks)

At O and O;

$$3x^2 = 15x \text{ (Point of intersection)}$$

$$3x^2 - 15x = 0$$

$$3x(x - 5) = 0$$

$$3x = 0$$

$$x = 0$$

$$x - 5 = 0$$

$$x = 5$$

when $x = 5$;

$$y = 15(5)$$

$$= 75$$

$$\therefore P(5, 75).$$

(b) By integration, determine the area of the shaded region. (3mks)

Area of the shaded;

$$= \int_{x=0}^{5} (15x)\,dx - \int_{x=0}^{5} (3x^2)\,dx$$

$$= \left[\frac{15}{2}x^2\right]_0^5 - [x^3]_0^5$$

$$= \left[\left(\frac{15}{2}(5)^2\right) - \left(\frac{15}{2}(0)^2\right)\right] - [5^3 - 0^3]$$

$$= 187.5 - 125$$

$$= 62.5 \text{ sq. units.}$$

(c) Estimate the area of the shaded region using trapezoidal rule with 5 strips. (4mks)

Area under $y = 15x$;

$$h = \frac{5-0}{5}$$

$$= 1 \text{ unit.}$$

x	0	1	2	3	4	5
y	0	15	30	45	60	75

Area;

$$= \frac{1}{2}[(0+75) + 2(15+30+45+60)]$$

$$= \frac{1}{2}[75 + 300]$$

$$= 187.5 \text{ square units}$$

Area under $y = 3x^2$.

x	0	1	2	3	4	5
y	0	3	12	27	45	75

Area;

$$= \frac{1}{2}[(0+75) + 2(3+12+27+45)]$$

$$= \frac{1}{2}[75 + 180]$$

$$= 127.5 \text{ sq. units}$$

Area of the shaded region;

$$= 187.5 - 127.5$$

$$= 60 \text{ sq. units.}$$

2. (a) complete the table below for the function $y = x^2 - 3x + 6$ in the range $-2 \leq x \leq 8$. (2mks)

x	−2	−1	0	1	2	3	4	5	6	7	8
y	16	10	6	4	4	6	10	16	24	34	46

(b) Use the trapezium rule with 5 strips to estimate the area bounded by the curve $y = x^2 - 3x + 6$ in the range $-2 \leq x \leq 8$ and the $x-$axis. (3mks)

$$h = \frac{8-(-2)}{5}$$

$$= 2$$

$$A = \frac{2}{2}\{(16+46) - 2(6+4+10+24)\}$$

$$= 1\{62+88\}$$

$$= 150 \text{ sq. units.}$$

(c) Use the mid-ordinate with 5 strips to estimate the area bounded by the curve $y = x^2 - 3x + 6$ in the range $-2 \leq x \leq 8$ and the $x-$axis. (2mks)

$$h = \frac{8-(-2)}{5}$$

$$= 2$$

$$A = 2(10+4+6+6+16+34)$$

$$= 140 \text{ sq. units.}$$

(d) By integration, determine the actual area bounded by the curve $y = x^2 - 3x + 6$ in the range $-2 \leq x \leq 8$ and the $x-$axis. (3mks)

$$A = \int_{x=-2}^{8} (x^2 - 3x + 6)\, dx$$

$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 6x\right]_{-2}^{8}$$

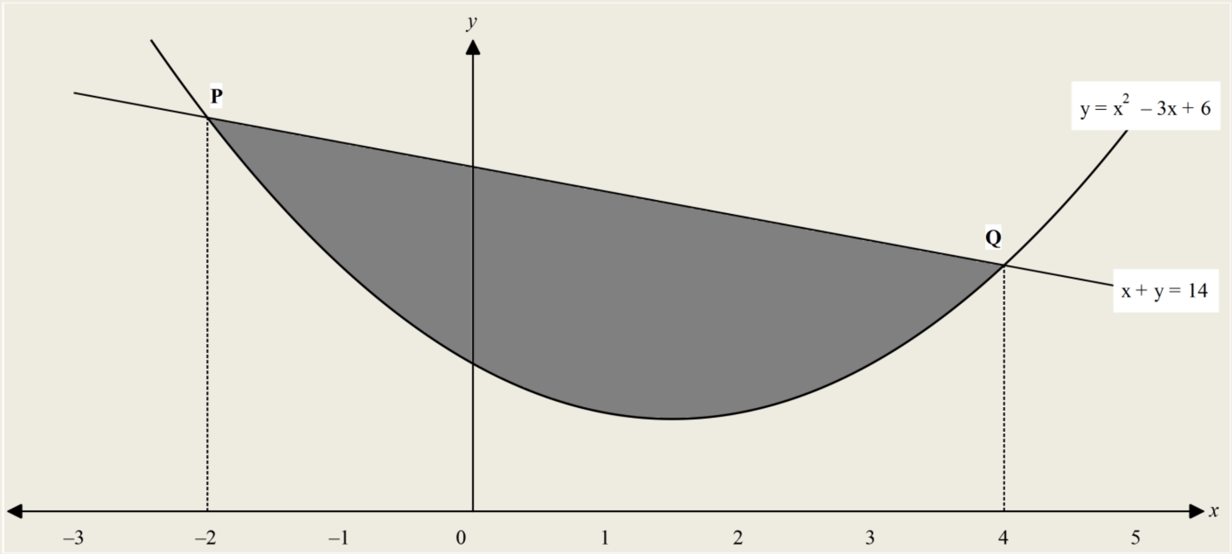
$$= \left[\frac{1}{3}(8)^3 - \frac{3}{2}(8)^2 + 6(8)\right] - \left[\frac{1}{3}(-2)^3 - \frac{3}{2}(-2)^2 + 6(-2)\right]$$

$$= \left[\frac{368}{3}\right] - \left[-\frac{62}{3}\right]$$

$$= \frac{430}{3}$$

$$= 143\frac{1}{3} \text{ sq. units.}$$

3. The diagram shows a sketch of a curve $y = x^2 - 3x + 6$ intersecting with the line $x + y = 14$ for $-2 \leq x \leq 4$ at points P and Q.



(a) Find the coordinates of points P and Q. (1mk)

At points of intersection;

$\mathbf{x^2 - 3x + 6 = -x + 14}$

$\mathbf{At\ P, x = -2}$

$\mathbf{y = -(-2) + 14}$

$\mathbf{= 16}$

$\mathbf{P(-2, 16)}$

At Q,

$\mathbf{X = 4}$

$\mathbf{Y = -4 + 14}$

$\mathbf{= 10}$

$\mathbf{Q(4, 10).}$

(b) Fill the table below for the values of y for $y = x^2 - 3x + 6$. (2mks)

x	−2	−1	0	1	2	3	4
y	16	10	6	4	4	6	10

(c) Determine the area bounded by the curve $y = x^2 - 3x + 6$ and the line $x + y = 14$, using trapezium rule with 6 strips. (4mks)

area of $y = x^2 - 3x + 6$

x	−2	−1	0	1	2	3	4
y	16	10	6	4	4	6	10

$$h = \frac{4-(-2)}{6}$$

$$= 1$$

$$A = \frac{1}{2}[(16+10) + 2(10+6+4+4+6)]$$

$$= \frac{1}{2}(26+60)$$

$$= 43 \text{ sq. unit}$$

Area of $y = -x + 14$.

x	−2	−1	0	1	2	3	4
y	16	15	14	13	12	11	10

$$A = \frac{1}{2}[(16+10) + 2(15+14+13+12+11)]$$

$$= \frac{1}{2}(26+130)$$

$$= 78 \text{ sq. unit}$$

Shaded area;

$$= 78 - 43$$

$$= 35 \text{ sq. units.}$$

(d) Calculate the exact area of the shaded region. (3mks)

Area under $y = 14 - x$

$$A = \int_{x=-2}^{4} (14 - x)dx$$

$$= \left[14x - \frac{1}{2}x^2\right]_{-2}^{4}$$

$$= \left[14(4) - \frac{1}{2}(4)^2\right] - \left[14(-2) - \frac{1}{2}(-2)^2\right]$$

$$= 48 - (-30)$$

$$= 78 \text{ sq. units}$$

Area under $y = x^2 - 3x + 6$

$$A = \int_{x=-2}^{4} (x^2 - 3x + 6)dx$$

$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 6x\right]_{-2}^{4}$$

$$= \left[\frac{1}{3}(4)^3 - \frac{3}{2}(4)^2 + 6(4)\right] - \left[\frac{1}{3}(-2)^3 - \frac{3}{2}(-2)^2 + 6(-2)\right]$$

$$= \frac{64}{3} - \left[-\frac{62}{3}\right]$$

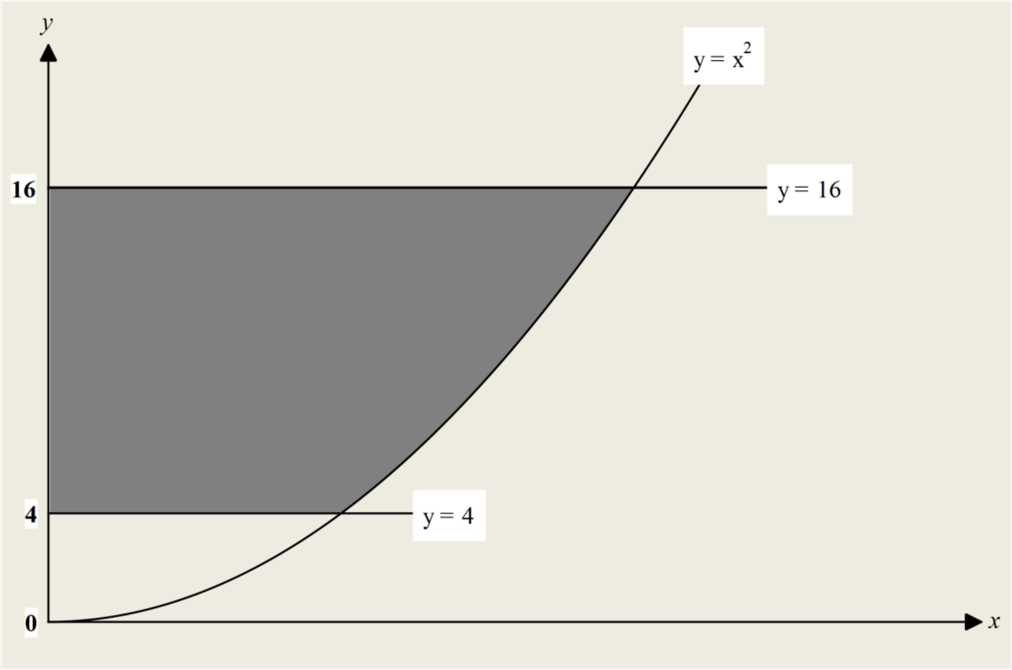
$$= 42 \text{ sq. units}$$

Area of shaded region;

$$= 78 - 42$$

$$= 36 \text{ sq. units.}$$

4. The shaded region in the figure below is bounded by the curve $y = x^2$ and the line $y = 4$ and $y = 16$.



(a) Calculate the exact area of the shaded region. (4mks)

$$y = x^2$$

$$x = \sqrt{y}$$

$$x = y^{\frac{1}{2}}$$

Area of shaded;

$$= \int_{y=4}^{16} \left(y^{\frac{1}{2}}\right) dy$$

$$= \left[\frac{2}{3} y^{\frac{3}{2}}\right]_4^{16}$$

$$= \left[\frac{2}{3}(16)^{\frac{3}{2}}\right] - \left(\frac{2}{3}(4)^{\frac{3}{2}}\right)$$

$$= 42\frac{2}{3} - 5\frac{1}{3}$$

$$= 37\frac{1}{3} \text{ sq. units.}$$

(b) Estimate the area of the shaded region using;

i. Trapezium rule and a height of 1 unit. (3mks)

$\mathbf{At\ P, y = 4}$

$\mathbf{4 = x^2}$

$\mathbf{x = \sqrt{4}}$

$\mathbf{x = \pm 2}$

$\mathbf{at\ Q, y = 16}$

$\mathbf{x^2 = 16}$

$\mathbf{x = \sqrt{16}}$

$\mathbf{x = \pm 4}$

$\mathbf{Area\ under\ line\ y}$

$\mathbf{= 16 - Area\ \ under\ line\ y}$

$\mathbf{= 4.}$

x	0	1	2
y_1	16	16	16
y_2	4	4	4
$y_1 - y_2$	12	12	12

$$\mathbf{Area = \frac{1}{2}[(12 + 12) + 2(12)]}$$

$$\mathbf{= 24\ sq.units}$$

$\mathbf{Area\ from\ 2 \leq x \leq 4.}$

x	2	3	4
y_1	16	16	16
y_2	4	9	16
$y_1 - y_2$	12	7	0

$$\mathbf{Area = \frac{1}{2}[(12 + 0) + 2(7)]}$$

$$\mathbf{= 13\ sq.units}$$

$\mathbf{Total\ Area\ of\ shaded\ region;}$

$$\mathbf{= 24 + 13}$$

$$\mathbf{= 37\ sq.units.}$$

ii. Mid – ordinate rule and height of 1 unit. (3mks)

Area between lines;

$x = 16$

and $y = 4$

from

$0 \leq x \leq 2.$

x	0.5	1.5
y_1	16	16
y_2	4	4
$y_1 - y_2$	12	12

$\text{Area} = 1(12 + 12)$

$= 24 \text{ sq. units.}$

Area between lines;

$y = 16 \text{ and } y = x^2$

from $2 \leq x \leq 4.$

x	2.5	3.5
y_1	16	16
y_2	6.25	12.25
$y_1 - y_2$	9.75	3.75

$\text{Area} = 1(9.75 + 3.75)$

$= 13.5 \text{ sq. units}$

Total shaded area;

$= 24 + 13.5$

$= 37\frac{1}{2} \text{ sq. units.}$

5. (a) complete the table below for $y = 3\sin 2x$ in the range $0^c \le x \le \frac{\pi^c}{2}$ to 4 significant figures.

x^c	0	$\frac{1}{12}\pi$	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{5}{12}\pi$	$\frac{1}{2}\pi$
y	0.0000	1.5000	2.5981	3.0000	2.5981	1.5000	0.0000

(b) Estimate the area of bounded by the curve $y = 3\sin 2x$ for $0^c \le x \le \frac{\pi^c}{2}$ using;

i. Trapezium rule and 7 ordinates. (3mks)

No. of strips;

$$= 7 - 1$$

$$= 6$$

$$h = \frac{\frac{1}{2}\pi^c - 0}{6}$$

$$= \frac{1}{12}\pi^c$$

$$A = \frac{1}{24}\pi^c[(0+0) + 2(1.5 + 2.598 + 3 + 2.598 + 1.5)]$$

$$= \frac{1}{24}\pi^c(22.392)$$

$$= 0.933\pi^c$$

$$= 2.9311 \text{ sq. units.}$$

ii. Mid – ordinate rule and 3 strips. (3mks)

$$h = \frac{\frac{1}{2}\pi^c - 0}{3}$$

$$= \frac{1}{6}\pi^c$$

$$A = \frac{1}{6}\pi^c(1.5 + 3 + 1.5)$$

$$= \frac{1}{6}\pi^c \times 6$$

$$= \pi^c$$

$$= 3.1416 \text{ sq. units.}$$

(c) Given that $\int_0^{\frac{1}{2}\pi}(3\sin 2x)dx = 3$, calculate the error in (b)(i) and (ii) above. (2mks)

Actual area;

$$\int_0^{\frac{1}{2}\pi}(3\sin 2x)dx = 3$$

$$\text{Error} = \frac{3 - 2.9311}{3}$$

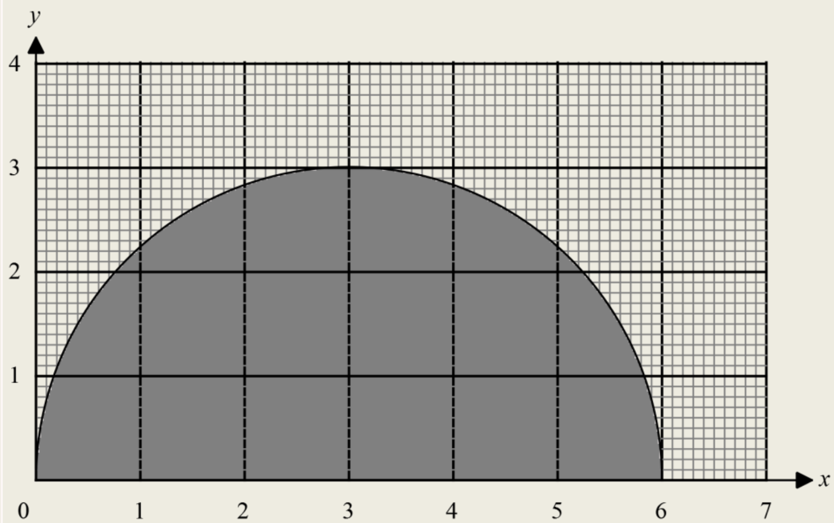
$$= 0.02297$$

or

$$\text{Error} = \frac{3 - 3.1416}{3}$$

$$= 0.0472$$

6. The figure below shows a semi – circle centre (3,0) and radius 3 units.



(a) Estimate the area of the semi – circle using;

i. Trapezoidal rule and 6 strips. (4mks)

x	0	1	2	3	4	5	6
y	0	2.2	2.8	3	2.8	2.2	0

$$\mathbf{h = \frac{6-0}{6}}$$

$$\mathbf{= 1\ unit}$$

$$\mathbf{\frac{1}{2}[(0+) + 2(2.2 + 2.8 + 3 + 2.8 + 2.2)]}$$

$$\mathbf{= \frac{1}{2}(26)}$$

$$\mathbf{= 13\ sq.units.}$$

ii. Mid – ordinate rule and 6 strips. (4mks)

$$\mathbf{h = \frac{6-0}{6}}$$

$$\mathbf{= 1\ unit}$$

x	0.5	1.5	2.5	3.5	4.5	5.5
y	1.6	2.6	2.95	2.95	2.6	1.6

$$\mathbf{A = 1(1.6 + 2.6 + 2.95 + 2.95 + 2.6 + 1.6)}$$

$$\mathbf{= 14.3\ sq.units.}$$

(b) Find, in terms of π, the error in the area of the semi − circle when mid − ordinate rule is used as in (a) (ii) above. (2mks)

Exact Area;

$$A = \frac{1}{2}\pi r^2$$

$$= \frac{\pi(3)^2}{2}$$

$$= \frac{9}{2}\pi$$

Error absolute;

$$= \left|\frac{9}{2}\pi - 14.3\right|.$$

7. A region **R** is bounded by the curve $y = x^3$, the x − axis and the ordinates $x = -3$ and $x = 3$. Determine;

(a) The exact area of the region **R**. (4mks)

Points where the curve cuts x − axis;

$$y = 0$$

$$x = 0$$

so we integrate from;

−3 to 0

then from 0 to 3

$$A = 2 \times \int_{-3}^{0} (x^3)dx$$

$$= 2 \times \left[\frac{1}{4}y^4\right]_{-3}^{0}$$

$$= 2\left\{\left[\frac{1}{4}(0)^4\right] - \left[\frac{1}{4}(-3)^4\right]\right\}$$

$$= 2 \times 20\frac{1}{4}$$

$$= 40\frac{1}{2} \text{ sq. units.}$$

(b) Estimate the area of the region **R** using the mid − ordinate rule and 6 ordinates. (4mks)

$$h = \frac{3-(-3)}{6}$$

$$= 1 \text{ unit.}$$

x	−2.5	−1.5	−0.5	0.5	1.5	2.5
y	−15.625	−3.375	−0.125	0.125	3.375	15.625

$$A = 1(15.626 + 3.375 + 0.125 + 0.125 + 3.375 + 15.625)$$

$$= 38\frac{1}{4} \text{ sq. units.}$$

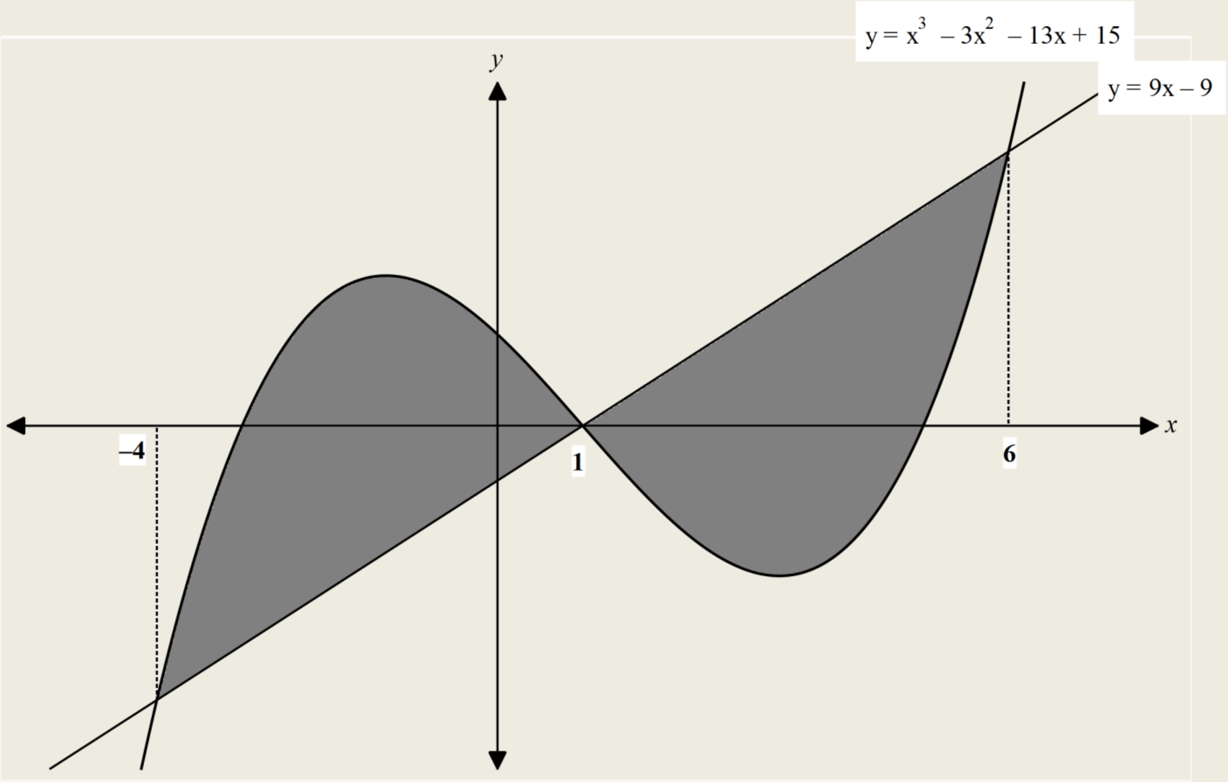
(c) Calculate the percentage error in the area of **R** in (b) above. (2mks)

% error;

$$= \left(\frac{40\frac{1}{2} - 38\frac{1}{4}}{40\frac{1}{2}}\right)100$$

$$= 5\frac{5}{9}\%.$$

8. In the figure below, the shaded region is bounded by the curve $y = x^3 - 3x^2 - 13x + 15$ and the straight line $y = 9x - 9$.



(a) Calculate the exact area of the shaded region. (4mks)

$$= 2\left\{\int_1^6 (9x - 9)\,dx - \int_1^6 (x^3 - 3x^2 - 13x + 15)\,dx\right\}$$

$$= 2\left\{\left[\frac{9}{2}x^2 - 9x\right]_{-3}^{0} + \left[\frac{x^4}{4} - x^3 - \frac{13}{2}x^2 + 15x\right]_{-3}^{0}\right\}$$

$$= 2\left\{\frac{225}{2} - \left(-\frac{175}{4}\right)\right\}$$

$$= 312.5 \text{ sq. units.}$$

(b) Estimate the area of the shaded region using trapezium rule with 10 strips. (4mks)

Consider from intersection point towards R. H. S.

For the curve $y = x^3 - 3x^2 - 13x + 15$.

x	1	2	3	4	5	6
y	0	−15	−24	−21	0	0

Consider from intersection point towards R. H. S.

For the curve $y = 9x - 9$.

x	1	2	3	4	5
y	0	9	18	27	36

Taking absolute value for y_T

$$y_1 = (0 + 0)$$

$$= 0$$

$$y_2 = (15 + 9)$$

$$= 24$$

$$y_3 = (24 + 18)$$

$$= 42$$

$$y_4 = (21 + 27)$$

$$= 48$$

$$y_5 = (0 + 36)$$

$$= 36$$

$$y_6 = 0$$

$$A = 2\left[\frac{1}{2}\{(0 + 0) + 2(24 + 42 + 48 + 36)\}\right]$$

$$= 2[0 + 150]$$

$$= 300 \text{ sq. units.}$$

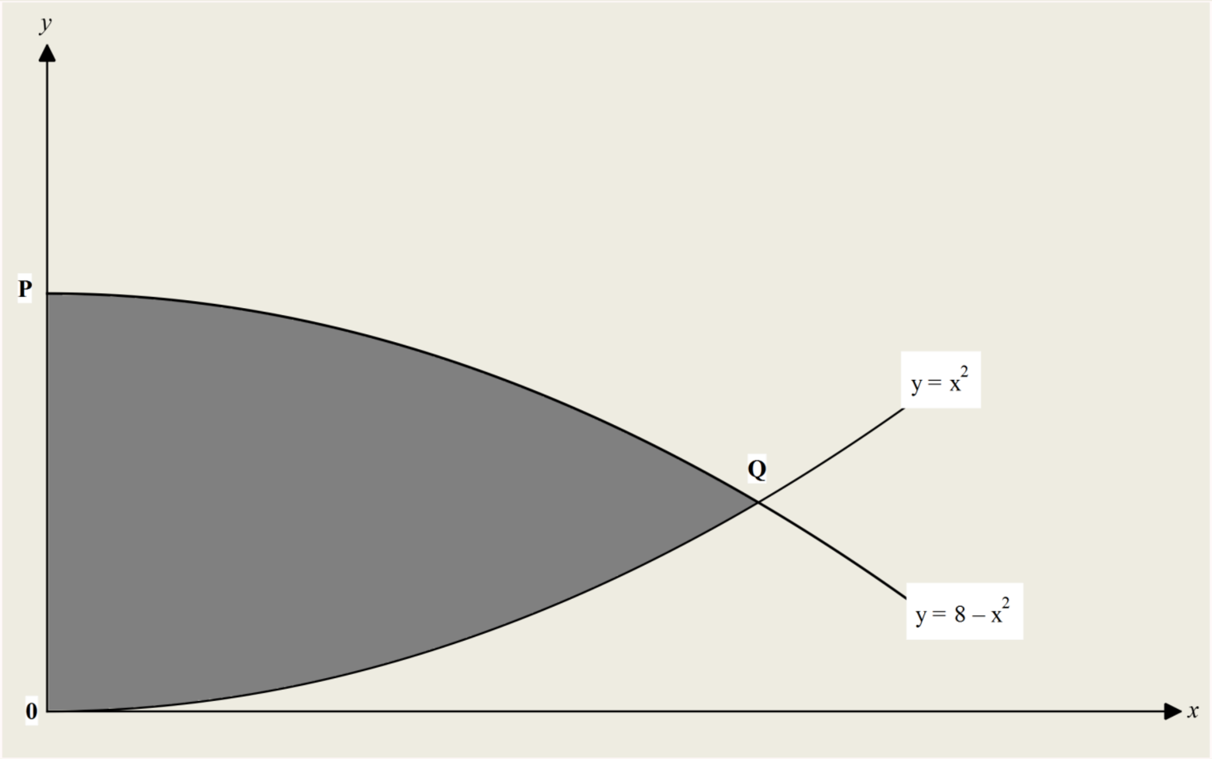
(c) Hence determine the percentage error in the area in (b) above. (2mks)

% error;

$$= \left(\frac{312.5 - 300}{312.5}\right)100$$

$$= 4\%.$$

9. In the figure below, the curve $y = x^2$ and $y = 8 - x^2$ intersect at Q.



(a) Determine the coordinates of Q. (1mk)

At point of intersection, Q;

$$\mathbf{x^2 = 8 - x^2}$$

$$\mathbf{2x^2 = 8}$$

$$\mathbf{x^2 = 4}$$

$$\mathbf{x = \sqrt{4}}$$

$$\mathbf{x = \pm 2}$$

Point:

$$\mathbf{y = 8 - (2)^2}$$

$$\mathbf{Q(2, 4)}$$

(b) By integration, calculate the area of the shaded region. (3mks)

area under curve;

$$y = 8 - x^2$$

$$A = \int_0^2 (8 - x^2)dx$$

$$= \left[8x - \frac{1}{3}x^3\right]_0^2$$

$$= \left[8(2) - \frac{1}{3}(2)^3\right] - \left[8(0) - \frac{1}{3}(0)^3\right]$$

$$= \frac{40}{3} - 0$$

$$= \frac{40}{3}$$

Area under the curve;

$$y = x^2$$

$$= \int_0^2 (x^2)\,dx$$

$$= \left[\frac{1}{3}x^3\right]_0^2$$

$$= \left[\frac{1}{3}(2)^3\right] - \left[\frac{1}{3}(0)^3\right]$$

$$= \frac{8}{3}$$

Area of the shaded region;

$$= \frac{40}{3} - \frac{8}{3}$$

$$= 10\frac{2}{3} \text{ sq. units.}$$

(c) Estimate the area of the shaded region using;

i. Trapezium rule with 4 strips. (3mks)

$$h = \frac{2 - 0}{4}$$

$$= 0.5 \text{ units.}$$

x	0	0.5	1	1.5	2
$y_1 = 8 - x^2$	8	7.75	7	5.75	4
$y_2 = x^2$	0	0.25	1	2.25	4

Area under the curve;

$$y = 8 - x^2$$

$$= \frac{0.5}{2}[(8 + 4) + 2(7.75 + 7 + 5.75)]$$

$$= 13.25 \text{ sq. units}$$

Area under the curve;

$$y = x^2$$

$$= \frac{0.5}{2}[(0 + 4) + 2(0.25 + 1 + 2.25)]$$

$$= 2.75 \text{ sq. units}$$

Area of the shaded region;

$$= 13.25 - 2.75$$

$$= 10\frac{1}{2} \text{ sq. units.}$$

ii. Mid – ordinate rule with 4 strips. (3mks)

$$h = \frac{2-0}{4}$$

$$= 0.5 \text{ units}$$

x	0.25	0.75	1.25	1.75
$y_1 = 8 - x^2$	7.9375	7.4375	6.4375	4.9375
$y_2 = x^2$	0.0625	0.5625	1.5625	3.0625

Area under curve;

$$y = 8 - x^2$$

$$= 0.5(7.9375 + 7.4375 + 6.4375 + 4.9373)$$

$$= 0.5 \times 26.75$$

$$= 13.375 \text{ sq. units}$$

Area under curve;

$$y = x^2$$

$$= 0.5(0.0625 + 0.5625 + 1.5625 + 3.0625)$$

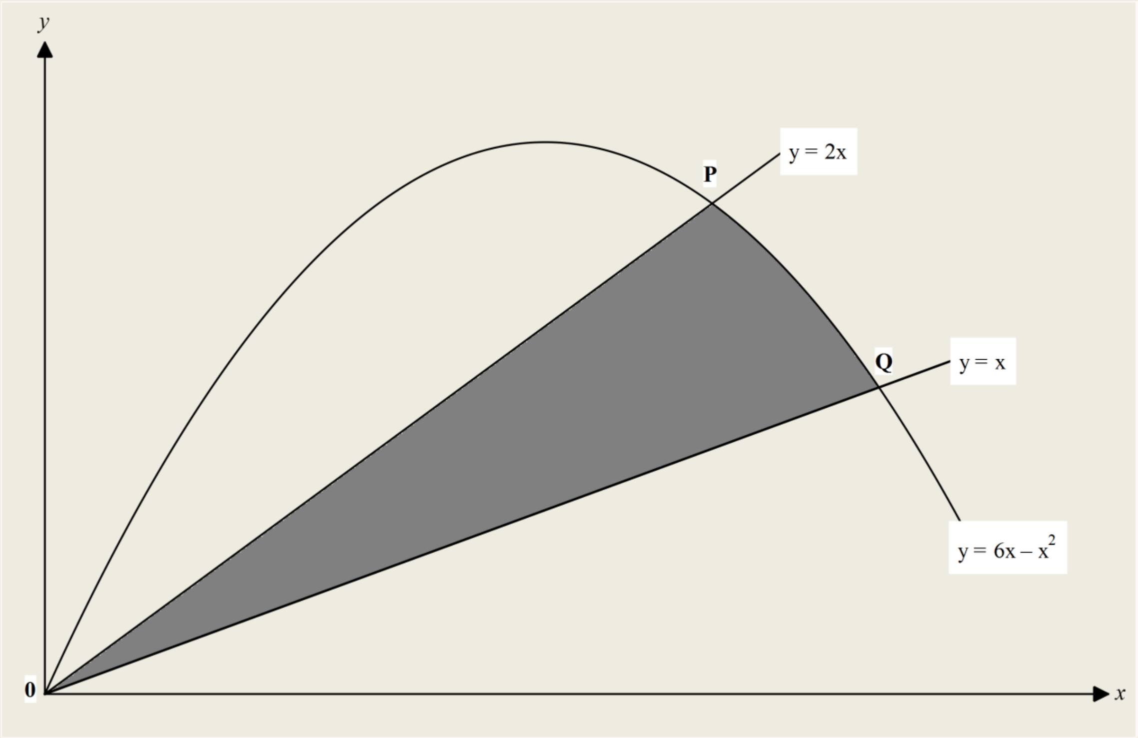
$$= 2.625 \text{ sq. units}$$

Area of the shaded region;

$$= 13.375 - 2.625$$

$$= 10\frac{3}{4} \text{ sq. units.}$$

10. In the figure below, the shaded region is bounded by the lines $y = 2x, y = x$ and the curve $y = 6x - x^2$. He two straight lines intersect the curve at points P and Q respectively.



(a) Determine the coordinates of points P and Q. (2mks)

Points of intersection;

$6x - x^2 = 2x$

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$

$x = 4$

at $x = 4$;

$y = 2(4)$

$= 8$

$P(4, 8)$

Also;

$6x - x^2 = x$

$x^2 - 5x = 0$

$x(x - 5) = 0$

$x = 0$

$x = 5$

at $x = 5$;

$y = 5$

$Q(5, 5)$

(b) Calculate the exact area of the shaded region. (4mks)

Area under the curve;

$$y = 6x - x^2$$

$$= \int_0^5 (6x - x^2)\,dx$$

$$= \left[3x^2 - \frac{1}{3}x^3\right]_0^5$$

$$= \left[3(5)^2 - \frac{1}{3}(5)^3\right] - \left[3(0)^2 - \frac{1}{3}(0)^3\right]$$

$$= 33\frac{1}{3} - 0$$

$$= 33\frac{1}{3}\text{ sq. units}$$

Area under line;

$$y = x$$

$$= \int_0^5 (x)\,dx$$

$$= \left[\frac{1}{2}x^2\right]_0^5$$

$$= \left[\frac{1}{2}(5)^2\right] - \left[\frac{1}{2}(0)^2\right]$$

$$= 12.5\text{ sq. units}$$

Area under line;

$$y = 2x$$

$$== \int_0^4 (2x)\,dx$$

$$\left[x^2\right]_0^4$$

$$= \left[(4)^2\right] - \left[(0)^2\right]$$

$$= 16\text{ sq. units}$$

Area under curve;

$$y = 6x - x^2$$

$$= \int_0^4 (6x - x^2)\,dx$$

$$= \left[3x^2 - \frac{1}{3}x^3\right]_0^4$$

$$= \left[3(4)^2 - \frac{1}{3}(4)^3\right] - \left[3(0)^2 - \frac{1}{3}(0)^3\right]$$

$$= 26\frac{2}{3} - 0$$

$$= 26\frac{2}{3}\text{ sq. units}$$

Area bounded by $y = 6x - x^2$ and $y = 2x$;

$$= 26\frac{2}{3} - 16$$

$$= 10\frac{2}{3}\text{ sq. units}$$

Area bounded by $y = 6x - x^2$ and $y = x$;

$$= 33\frac{1}{3} - 12\frac{1}{2}$$

$$= 20\frac{5}{6}$$

Area of the shaded region;

$$= 20\frac{5}{6} - 10\frac{2}{3}$$

$$= 10\frac{1}{6}.$$

(c) Taking the height of each trapezium as 1 unit, estimate the area of the shaded region. (4mks)

x	0	1	2	3	4
$y_1 = 6x - x^2$	0	5	8	9	8
$y_2 = 2x$	0	2	4	6	8

Area of curve;

$$y = 6x - x^2$$

$$= \frac{1}{2}[(0+8) + 2(5+8+9)]$$

$$= 26 \text{ sq. units}$$

Are of the curve;

$$y = 2x$$

$$\frac{1}{2}[(0+8) + 2(2+4+6)]$$

$$= 16 \text{ sq. units}$$

Area bounded by $y = 6x - x^2$ **and** $y = 2x$;

$$= 26 - 16$$

$$= 10 \text{ sq. units}$$

For $y = 6x - x^2$ **and** $y = x$

x	0	1	2	3	4	5
$y_1 = 6x - x^2$	0	5	8	9	8	5
$y_2 = 2x$	0	1	2	3	4	5

Area of curve;

$$y = 6x - x^2$$

$$= \frac{1}{2}[(0+5) + 2(5+8+9+8)]$$

$$= 32.5 \text{ sq. units}$$

Are of the curve;

$$y = x$$

$$\frac{1}{2}[(0+5) + 2(1+2+3+4)]$$

$$= 12.5 \text{ sq. units}$$

Area bounded by $y = 6x - x^2$ **and** $y = x$;

$$= 32.5 - 12.5$$

$$= 20 \text{ sq. units}$$

Area of the shaded region;

$$= 20 - 10$$

$$= 10 \text{ sq. units.}$$

Differentiation and Integration.

1.